

The Cooper pair from radio-frequency excitations in ultracold gases

J. Dukelsky¹ and G. Ortiz²

¹ *Instituto de Estructura de la Materia, CSIC, Serrano 123, 28006 Madrid, Spain*

² *Department of Physics, Indiana University, Bloomington, IN 47405, USA*

We discuss the concept of Cooper pair in the context of recent experimental studies of radio-frequency excitations in ultracold atomic gases. We argue that the threshold energy determines the size of the Cooper pair emergent from the exact solution of the reduced BCS problem, and elaborate on the physical distinction between bosonic and fermionic Cooper pairs.

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A few years ago [1] we thoroughly analyzed the notion and nature of Cooper pairs in a correlated Fermi system. As shown by Cooper, an isolated pair in an inert Fermi sea constitutes a true bound state while a correlated pair in a superconductor or superfluid medium behaves as a quasi-resonance. Simply said, the single pair case corresponds to a two-particle problem while the latter represents a true many-body problem. We also raised the provocative question *What is a Cooper pair?*, and argued that the Cooper pair in a correlated Fermi superfluid is a statistical concept that acquires physical significance only through the particular experimental probe one uses to test the system. For example, if one measures the antiparallel spin part of the static density-density correlation function of the Fermi system, one is probing the (Fourier transformed) pair wavefunction $u_{\mathbf{k}}v_{\mathbf{k}}$.

Recent experimental studies of radio-frequency (RF) excitations in ultracold Fermi gases do not measure the latter pair wavefunction. Indeed, by using a mean-field BCS description of the system one may conclude that they do probe the BCS pair wavefunction $v_{\mathbf{k}}/u_{\mathbf{k}}$. But, what property of this pair wavefunction do they really probe? A main goal of this short Note is to argue that what these experiments really probe are the properties of the Cooper pair concept that emerges from the exact solution of the reduced BCS problem. For example, we show that the pair size derived from the threshold energy as recently proposed in Ref. [2] is nothing more than the notion of pair size defined in Ref. [1].

The reduced s-wave paring BCS Hamiltonian, for fermions enclosed in a volume V and with non-interacting dispersion $\varepsilon_{\mathbf{k}}$, can be written as

$$H = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} n_{\mathbf{k}} + \frac{G}{V} \sum_{\mathbf{k}, \mathbf{k}'} c_{\mathbf{k}\uparrow}^{\dagger} c_{-\mathbf{k}\downarrow}^{\dagger} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}, \quad (1)$$

which includes all terms with time-reversed pairs ($\mathbf{k} \uparrow, -\mathbf{k} \downarrow$) from a contact interaction between fermions of mass m , momentum \mathbf{k} , and spin component $\sigma = \uparrow, \downarrow$ (represented by the (creation) operator $c_{\mathbf{k}\sigma}^{\dagger}$). In first quantization language, the resulting *exact* BCS state [3] for $M = N/2$ pairs is given by ($x_j = (\mathbf{r}_j, \sigma_j)$)

$$\Psi(x_1, \dots, x_N) = \mathcal{A}[\phi_1(x_1, x_2) \cdots \phi_M(x_{N-1}, x_N)] , \quad (2)$$

with \mathcal{A} the antisymmetrizer, and the *pair* state

$$\phi_{\alpha}(x_i, x_j) = e^{i\mathbf{q} \cdot \frac{(\mathbf{r}_i + \mathbf{r}_j)}{2}} \varphi_{\alpha}(\mathbf{r}_i - \mathbf{r}_j) \chi(\sigma_i, \sigma_j) , \quad (3)$$

where χ is the spin function, \mathbf{q} is the pair center-of-mass momentum, and $\varphi_{\alpha}(\mathbf{r})$ the internal (relative coordinate) wavefunction. The many-body wavefunction (2) describes a collection of M different pairs (3) which, depending upon the strength of the interaction between particles G , may represent either a quasimolecular resonant state or a scattering state (i.e., free fermions). The state (2) is the natural generalization of the Cooper-pair problem without an inert Fermi sea, and with all pairs subjected to the pairing interaction. Only in the extreme BEC limit this physical state and Leggett's ansatz [4], which assumes *all equal pairs*, are equivalent.

For a uniform 3D system in the thermodynamic limit the exact solution of the Hamiltonian (1) reduces to the BCS gap equation [5]. The singularity displayed by this equation can be regularized by introducing the scattering length a_s , resulting in a model that describes the BCS-to-BEC crossover in terms of a single parameter $\eta = 1/k_F a_s$, with k_F the Fermi momentum [4]. We studied the character and nature of these pairs in the BCS-to-BEC crossover region. Our results [1] suggested a scenario in which correlated pairs and free fermions coexist in the BCS region while there is a collection of quasibound molecules in the BEC region. More precisely, the pair wavefunction has the *universal* form [1]

$$\varphi_E(r) = \sqrt{\frac{m \operatorname{Im}(\sqrt{E})}{2\pi\hbar^2}} \frac{e^{-r\sqrt{-mE/\hbar^2}}}{r}, \quad (4)$$

independently of the value of η , where E is the spectral parameter or the pair energy determined from the Richardson (Bethe ansatz) equations [3]. It is important to emphasize that the many-body state (2) with the pair wavefunctions given by Eq. (4) is not an ansatz but it is the state that results from the exact solution to Eq. (1). The character of the pair wavefunction is determined by E which is, in general, a complex number. If E is real and positive then φ_E represents a scattering state; if it is complex with a positive real part it is a Cooper resonance, while if it is complex with a negative real part it is a quasibound molecular state. As a function of η , the distribution of pairs in the correlated state $\Psi(x_1, \dots, x_N)$

may have a different character, suggesting the following identifications

- **BCS regime:** a mixture of real E , and complex E with $\text{Re}(E) > 0$ (Cooper resonances and free fermions).

- **Pseudogap regime:** a mixture of complex E with both $\text{Re}(E) \gtrless 0$ (Cooper resonances and quasi-molecules).

- **BEC regime:** complex E with $\text{Re}(E) < 0$ (quasi-molecules).

The size of each pair, defined as the mean square radius $\xi_E = \sqrt{\langle \varphi_E | r^2 | \varphi_E \rangle}$, is given by

$$\xi_E^2 = \frac{\hbar^2}{2m (\text{Im} \sqrt{E})^2} = \frac{\hbar^2}{2m E_p} \quad (5)$$

where

$$2E_p = \sqrt{\text{Re}(E)^2 + \text{Im}(E)^2} - \text{Re}(E) \quad (6)$$

For a given value of the coupling η , the most correlated pairs are located close to the two extremes of a complex arc Γ with values $E = 2(\mu \pm i\Delta)$ (see Ref. [1, 5]). Inserting this value in the equation above leads to

$$E_p = \sqrt{\mu^2 + \Delta^2} - \mu, \quad (7)$$

In what follows we present results for some particular values of η

$$E_p = \begin{cases} \frac{\Delta^2}{2\epsilon_F} & \text{when } \eta \rightarrow -\infty \text{ (BCS)} \\ \frac{(1+x)\epsilon_F}{E\left(\frac{1-x}{2}\right)^{\frac{2}{3}}} & \text{when } \eta = 0 \text{ (resonance)} \\ \sqrt{2\eta}\epsilon_F & \text{when } \eta = \frac{8\pi^{2/3}}{\Gamma[\frac{1}{4}]^{8/3}} \text{ } (\mu = 0) \\ \frac{\hbar^2}{ma_s^2} & \text{when } \eta \rightarrow \infty \text{ (BEC)} \end{cases} \quad (8)$$

where $x \sim -0.65222953$ is the root of the Legendre function of the first kind $P_{\frac{1}{2}}$, i.e. $P_{\frac{1}{2}}(x) = 0$, and $E(y)$ is the complete elliptic integral of the second kind $((1+x)/E\left(\frac{1-x}{2}\right)^{\frac{2}{3}} \approx 0.314912)$. Figure 1 compares the size of this correlated pair with the sizes of the pair correlation function ξ_P , and the BCS wavefunction ξ_{BCS} . As already mentioned in [1] and recently suggested in [2], the size of the Cooper pair in the BCS region is *twice* the size of the pair correlation function. Moreover, while the size of the pair correlation function crosses the interparticle distance at $\eta \sim -0.5$ (where the fraction of the condensate is ~ 0.5), the Cooper pair size crosses the interparticle distance at resonance (where the fraction of the condensate approaches 1). The three wavefunctions coincide in the extreme BEC limit.

It is important to remark that the most correlated pair determines the minimum energy or threshold energy to

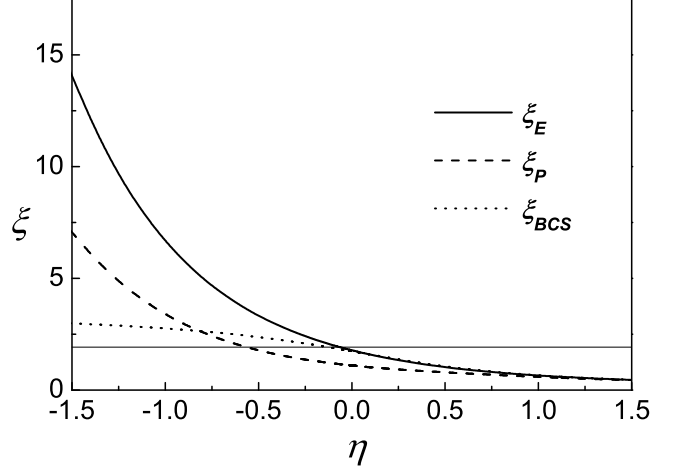


FIG. 1: Size of the Cooper pair ξ_E derived from the exact solution of the reduced BCS Hamiltonian of Eq. (1) in units of $1/k_F$. We have plotted sizes ξ_E , ξ_P , and ξ_{BCS} for the most correlated pair which corresponds to $E = 2(\mu \pm i\Delta)$ (solid line), the pair correlation function $u_{\mathbf{k}}v_{\mathbf{k}}$ (dashed line), and the BCS wavefunction $v_{\mathbf{k}}/u_{\mathbf{k}}$ (dotted line), respectively. The thin solid line is the interparticle distance $r_s = \sqrt{9\pi/4}$.

excite a particle in a RF-excitation experiment: $E_{th} = E_p$ in (7). In this respect, the operative definition $\xi_{th}^2 = \hbar^2/2mE_{th}$ used in [2] to extract the size of the Cooper pair from a fitting of the threshold energy represents a direct observation of the Cooper pair wavefunction (4) at the extremes of the arc for each value of η . E_{th} does not provide information about the size of either the pair wavefunctions $v_{\mathbf{k}}/u_{\mathbf{k}}$ or $u_{\mathbf{k}}v_{\mathbf{k}}$.

Finally, we would like to mention an essential difference between the structure of fermionic and bosonic atomic pairs. It has been shown that the Hamiltonian of Eq. (1) is exactly solvable for fermionic as well as for bosonic systems [7]. In the bosonic case the exact eigenstate is determined from the same Eqs. (2)-(4) with a symmetrizer in (2) instead of an antisymmetrizer. However, the bosonic ground state has always real and negative spectral parameters E for an arbitrary attractive interaction G . Physically, it represents a collection of *bound* diatomic molecules described by the Cooper pair wavefunction (4) in different excited states.

In summary, in this short Note we argued that RF excitation experiments probe the notion of Cooper pair emergent from the exact solution of the reduced BCS problem. We emphasize that the Cooper pair wavefunction (4) has a *universal form* as the RF dissociation spectrum in the whole crossover. In particular, the size derived from the threshold energy is directly related to the size of the Cooper pair defined in Ref. [1].

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